Review

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Initial Goal

- We began with the question of why output per capita differed around the world.
- Why are some countries rich and others poor?
- To answer this question, we concluded that we needed a model that was both complex enough to reflect reality and tractable enough to give us intuition.

Production Model

- We began with modeling production, where

$$Y = AK^{\alpha}L^{1-\alpha}.$$

- We said

$$\begin{aligned} A &: \text{ Total Factor Productivity} \\ K &: \text{ Capital} \\ L &: \text{ Labor} \\ \alpha &: \text{ Capital share } (0 < \alpha < 1) \end{aligned}$$

Production Model

- We then discussed the firm problem, which was given by

$$\max_{K,L} \; AK^{\alpha}L^{1-\alpha} - rK - wL$$

- The firm chose capital (K) and labor (L), taking the real wage (w) and interest rate (r) as given to maximize its profits.
- We discussed why this formulation of the problem with a CRS production function and marginal costs ensured zero profits.

Production Model

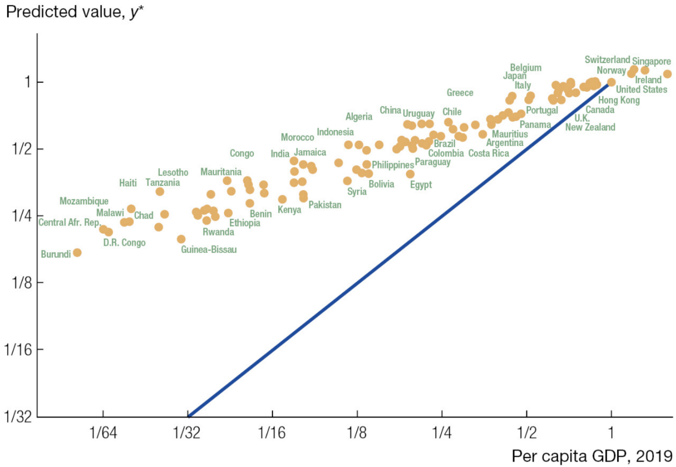
- We wanted to check to see if we could explain differences in per capita output by only physical capital (K).
- To do this, we set TFP (A) equal to one and plugged in capital per capita $(k \equiv K/L)$ into our production function to get output per capita $(y \equiv Y/L)$

$$\frac{Y}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L}$$
$$y = Ak^{\alpha}.$$

Production Model

FIGURE 4.5

The Model's Prediction for Per Capita GDP (U.S. = 1)



Production Model

- Our production model was unable to explain cross-country income differences.
- The model predicted poorer countries to be much richer than they actually are.
- We relaxed our assumption on TFP (A) and used it to fill in the gap between what our model predicted and what we saw in the data.
- We found that differences in physical capital explained around $1/3$ of the differences in output per capita around the world.

TFP

TABLE 4.4

Measuring TFP So the Model Fits Exactly

Country	Per capita GDP (y)	$\bar{k}^{1/3}$	Implied TFP ($\bar{A}$)
United States	1.000	1.000	1.000
Switzerland	1.203	1.142	1.054
U.K.	0.707	0.894	0.792
Japan	0.634	0.971	0.654
Italy	0.651	0.925	0.703
Spain	0.645	0.895	0.721
Brazil	0.233	0.580	0.402
South Africa	0.200	0.548	0.365
China	0.226	0.650	0.347
India	0.107	0.434	0.247
Burundi	0.013	0.172	0.074

Calculations are based on the equation $y = \bar{A}\bar{k}^{1/3}$. Implied productivity $\bar{A}$ is calculated from data on y and $\bar{k}$ for the year 2019, so that this equation holds exactly as $\bar{A} = y / \bar{k}^{1/3}$.

Solow Model

- At this point we had a model of production, but could not use it to explain the evolution of capital and output over time.
- The Solow model introduced us to the idea of capital accumulation

$$Y_t = AK_t^{\alpha}L^{1-\alpha}$$
$$K_{t+1} = s_K Y_t + (1-\delta)K_t.$$

- Capital in the next period was equal to the capital that did not depreciate ($(1-\delta)K_t$) and the output that was invested ($s_K Y_t$).

Solow Model

- With capital accumulation, we could see how capital changed over time.
- The change in capital between periods was given by

$$\Delta K_{t+1} = s_K Y_t - \delta K_t.$$

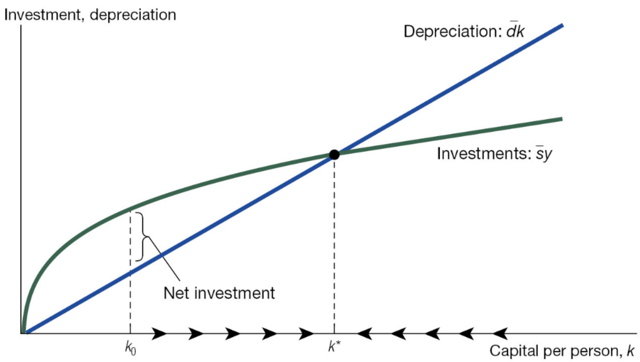
- In a steady state, where capital did not change between periods, we saw that investment had to equal depreciation.
- To change our equations to be per capita, we simply divided

$$\frac{\Delta K_{t+1}}{\bar{L}} = \frac{s_K Y_t - \delta K_t}{\bar{L}}$$
$$\Delta k_{t+1} = s_K y_t - \delta k_t.$$

Solow Model

FIGURE 5.1

The Solow Diagram



Solow and Romer

- In the Solow model, the investment rate of physical capital (s_K) and the physical capital depreciation rate (δ) drove steady state output per capita.
- The principle of transition dynamics stated that the further beneath a country was from its steady state output per capita, the faster it grew.
- This explained growth trajectories of countries not in their steady state, but the Solow model did not yield long-run growth in output per capita.
- For this, we gave more thought into TFP, calling A ideas with the Romer model.
- Ideas were non-rivalrous, which when incorporated into the model, gave us long-run growth in output per capita.

Romer

- In the Romer model we had output (Y_t), workers in the production sector (L_{yt}), workers in the ideas (L_{at}), the total number of workers (N_t), and ideas (A_t).

- These variables were governed by

$$
\begin{aligned}
L_{yt} + L_{at} &= N_t \\
Y_t &= A_t^{\gamma} L_{yt} \\
\Delta A_{t+1} &= \bar{z} L_{at} \\
\frac{\Delta N_{t+1}}{N_t} &= \bar{n} \\
L_{at} &= \bar{\ell} N_t
\end{aligned}
$$

- We assumed a constant share of workers in the ideas sector $(\bar{\ell})$ and that each worker generated $\bar{z}$ ideas.

Romer

- Since the model gave us long-run growth, there was no steady state.
- Instead, we looked for a balanced growth path, where output and the stock of ideas grew at constant rates.
- We found that the growth rate of output per capita was driven by the returns to scale parameter (γ) and population growth rate $(\bar{n})$.
- The fact ideas were non-rivalrous and thus benefited everyone gave us long-run growth.

Romer

- The Solow model did not give us long-run growth in output per capita.
- Incorporating ideas with the Romer model did, while adding physical capital gave us a growth rate of output per capita that was greater than the Romer model without physical capital.
- While physical capital did not drive long-run growth, physical capital enhanced long-run growth.

Human Capital

- In an effort to further explain why some countries are poor and others rich, we expanded our definition of capital to include human capital.
- The idea was that countries who have a more educated, higher skilled workforce can produce more with the same amount of labor.
- Investments were made into human capital in the form of education and on-the-job training.

Human Capital

- Using output (Y_t), physical capital (K_t), and human capital (H_t), we had

$$
\begin{aligned}
Y_t &= K_t^{\alpha}[(1 - s_H)H_t]^{1-\alpha} \\
K_{t+1} &= s_K Y_t + (1 - \delta)K_t \\
H_{t+1} &= (s_H H_t)^{\sigma} + (1 - \delta)H_t.
\end{aligned}
$$

- s_H was the share of time spent enhancing human capital and $(s_H H_t)^{\sigma}$ was education production.

Human capital

- Like the Solow model, the extended Solow model with human capital had a steady state.
- We found that the steady state for human capital was driven by the share of time spent enhancing human capital and human capital depreciation.
- We also found that the steady state for output per unit of human capital was influenced by both the share of time spent investing in human capital and the share of time spent in the production sector.
- Too much time spent investing in human capital and not in the production sector (or vice-versa) led to lower output per capita.

Unmeasured Capital

- We then generalized our notion of investment to include any payment now that give a return in the future.
- Doing this gave us "unmeasured capital," which included things such as R&D expenditures, software development, and human capital investment on the job.
- By expanding the definition of capital, we wanted to see if we could fully explain cross-country differences in output per capita.

Unmeasured Capital

- We modeled output (Y_t) as a function of physical capital (K_t), unmeasured capital (Z_t), and labor (L), which was augmented directly by TFP (A).

$$\begin{aligned} Y_t &= K_t^{\alpha_K} Z_t^{\alpha_Z} (AL)^{1-\alpha_K-\alpha_Z} \\ K_{t+1} &= (1-\delta_K)K_t + s_K Y_t \\ Z_{t+1} &= (1-\delta_Z)Z_t + s_Z Z_t. \end{aligned}$$

- We had separate investment rates for unmeasured capital (s_Z) and measured capital (s_K).

Steady State

- This model yielded a steady state as well.
- We used this to briefly test the ability of our model to compare differences between the United States and the world's poorest countries.
- We found that unmeasured capital was able to explain around 1/7th of the difference between the U.S. and the world's poorest countries.
- Even after extending the Solow model, we still had significant unexplained differences in output per capita across the world.

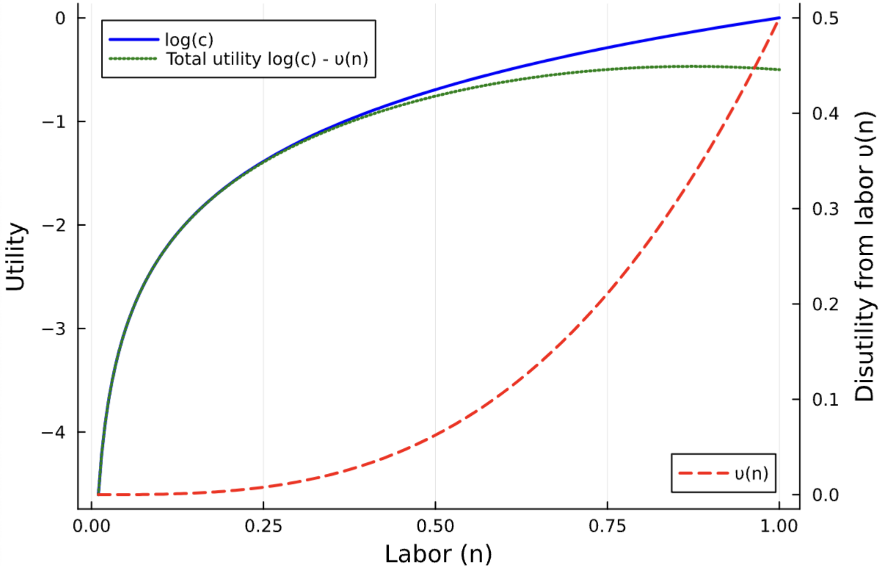
Institutional Differences

- We discussed several differences in institutions between countries that could be driving the differences in per capita output.
- Corruption and governance issues: Rampant corruption and stifling bureaucracies can lead to misallocated resources and discourage work.
- Political instability or conflict: War, civil unrest, and unpredictable regime changes are all events that cause fear and uncertainty, leading to reduced investment.
- Weak property rights: Fear of expropriation or an inability to enforce contracts can lead to reduced lending.
- Infrastructure and public goods: Poor public infrastructure and health can drive poor productivity.

Labor Supply

- Until this point we had assumed each worker supplied the same amount of labor in the economy.
- To endogenize the labor supply, we considered the labor supply decision of a representative household.
- The household worked in order to consume, while deriving utility from consumption and disutility from labor.
- The household had a tradeoff between leisure (time spent not working) and consumption, which it was able to afford from working.

Labor Supply



MRS

- In our model of the labor supply, we had

$$\max_{c,n} u(c) - v(n)$$

such that $c = wn$.

- We had a marginal rate of substitution, which was given by

$$-\frac{v'(n)}{u'(c)} = w.$$

- The change in disutility from an increase in labor had to be equal to the change in utility from additional consumption times the number of consumption units increasing labor yields (w).

Solving

- The MRS condition gave us an equation that the optimal consumption (c) and labor (n) combination solved.
- We needed to make sure our solution for consumption (c) and labor (n) was not only optimal, but affordable.
- Using the budget constraint, we had two equations and two unknowns, which allowed us to solve for labor supplied (n).

Labor Supply - Transfers

- We then looked at how government programs such as income taxes and transfers impact the labor supply decision.
- To incorporate these into our model, we assumed the representative household would pay $\tau_n\%$ of their labor income in a labor tax and receive T units of consumption in the form of transfers from the government.
- The household took the transfer (T) as given and independent from hours worked.
- The government had both tax revenue from the labor tax $(\tau_n wn)$ and expenditures from the transfer (T). These had to be equal in order for its budget constraint to be satisfied.

Labor Supply - Transfers

- We saw a model

$$\max_{c,n} \; c^{\alpha}(24-n)^{1-\alpha}$$
$$\text{such that } c = (1-\tau_n)wn + T$$

  where without a transfer the household always supplied 24α hours of labor, adding the transfer into the model reduced the level of labor supplied by the household.

- At each number of hours worked, the household was richer than the scenario with taxes but no transfer.

Equilibrium

- From our Production lecture we had a model of labor demanded.
- From our household model we had a model of labor supplied.
- We then discussed the idea of the labor market clearing, where the labor supplied in the economy equaled the labor demanded.
- The mechanism for this clearing was the real-wage, the price of labor, which adjusted to ensure that the labor market cleared.

Equilibrium: Household Side

- Our example representative household problem was

$$\max_{c,n} \; c - \frac{n^2}{2}$$
$$\text{such that } c = wn.$$

- We used the MRS to get

$$\begin{aligned} \text{MRS} &= w \\ -\frac{U_n}{U_c} &= w \\ n &= w. \end{aligned}$$

- We found that this problem implied the labor supplied by the household (n) as a function of the real-wage (w) was $L^s(w) = w$.

Equilibrium: Firm Side

- Our example representative firm problem was given by

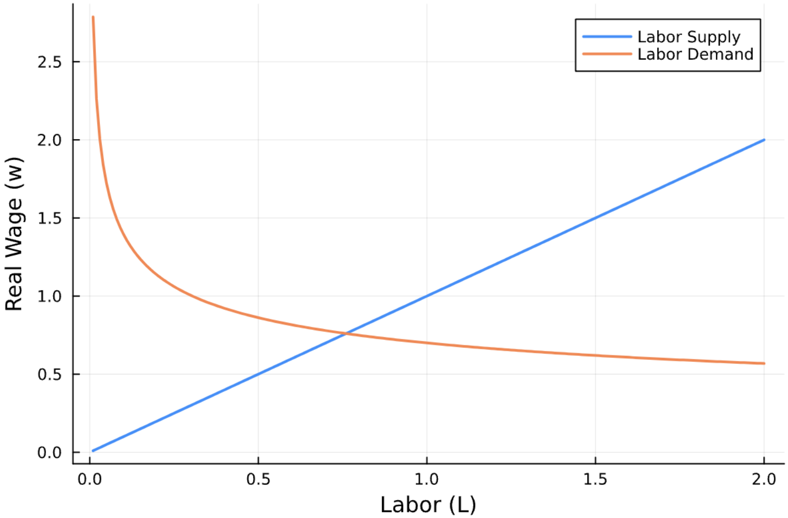
$$\max_{L} \; AL^{\theta} - wL.$$

- From our lecture on production we used the fact

$$\theta A L^{\theta-1} = w$$
$$L = \left(\frac{\theta A}{w}\right)^{\frac{1}{1-\theta}}.$$

- We found that the labor demanded by the firm (L) as a function of the real-wage (w) was given by $L^{d}(w) = \left(\frac{\theta A}{w}\right)^{\frac{1}{1-\theta}}$.

Equilibrium



Conclusion

- We modeled production, taking time to thoroughly think through TFP (A), capital (K), and labor (L).
- The Romer model gave us deeper insight into ideas (A).
- The Solow model and its extensions with human and unmeasured capital allowed us to think thoroughly about physical capital (K) and extensions.
- The labor supply model gave us tools to think more carefully about the forces driving observed labor in the economy (L).