Production[1]

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[1]All graphs are from our Jones textbook unless said otherwise

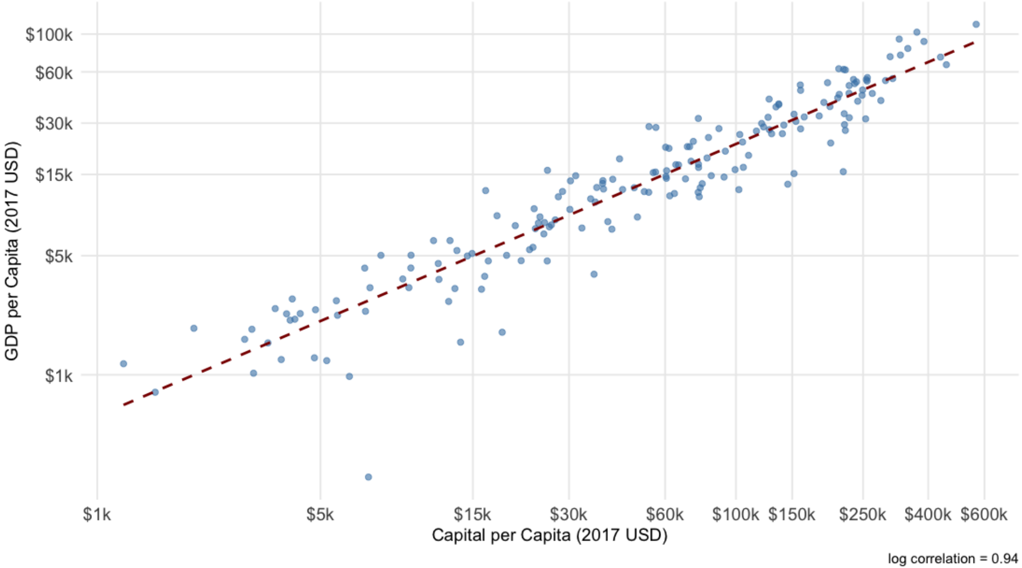
Review

- Last lecture we discussed the large differences in GDP per capita across the world.
- We saw GDP per capita was near zero for most of human history, growing only in the past few hundred years.
- We saw that in this recent time of growth, different countries had very different trajectories and historical averages of per capita gdp growth.
- We left with the plan to construct a model that explained why some countries are richer than others and what might be done enhance growth.

Motivation

- To better understand the vast differences in GDP per capita across countries, we need to model production.
- When comparing across countries, we used GDP per capita because output depends on the number of workers in a country.
  - The model of production we create will depend on the number of workers.
- What other factors would be essential to pair with labor in modeling production?
  - Total Factor Prouctivity - The productive efficiency of an economy. (The ability to make more with less)
  - Capital - Machines, warehouses, factories, etc.

Capital


$100k
$60k
$30k
$15k
$5k
$1k
GDP per Capita (2017 USD)
$1k
$5k
$15k
$30k
$60k
$100k
$150k
$250k
$400k
$600k
Capital per Capita (2017 USD)
log correlation = 0.94

Production

- To incorporate both capital and labor to model production, we will suggest a production function and check to see if its output matches what we see in the data.
- Our production function to model output (Y) will be

$$Y = F(K, L) = AK^{\alpha}L^{1-\alpha}.$$

$$\begin{aligned} A &: \text{ Total Factor Productivity (more on this later)} \\ K &: \text{ Capital} \\ L &: \text{ Labor} \\ \alpha &: \text{ Parameter governing contribution of capital and labor to output} \end{aligned}$$

- This is called the **Cobb-Douglas** production function, named after the economists Charles Cobb and Paul Douglas.

Output per person

- $F(K, L)$ tells us the output if using capital K and labor L.
- All of the comparisons from last lecture were done using output *per person*. To test our function, we will compare our production function's performance against output per person (y), or

$$
\begin{aligned}
y &\equiv \frac{Y}{L} \\
&= \frac{F(K, L)}{L} \\
&= \frac{AK^{\alpha}L^{1-\alpha}}{L} \\
&= AK^{\alpha}L^{-\alpha} \\
&= Ak^{\alpha}
\end{aligned}
$$

where $k \equiv \frac{K}{L}$ is capital per person.

Output per person

- We now have a measurement of output per person

$$y = Ak^{\alpha}$$

  where $k = \frac{K}{L}$ is capital per person.

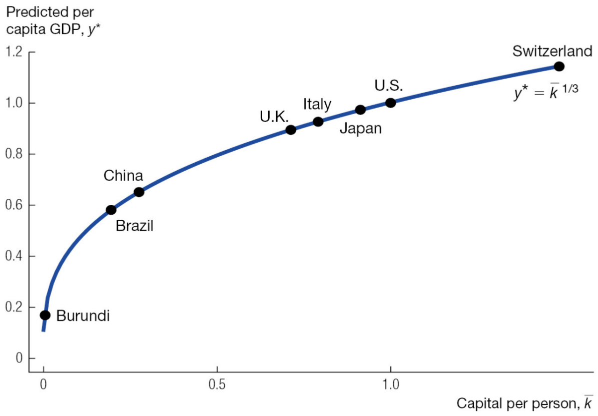
- We can use data on capital per person for a country, plug it into our function, and compare the output of our function with GDP per capita of a country.

- To do this, we'll need values for A and α. We will use $\alpha = 1/3$ (we'll justify this later) and try out $A = 1$.

Output per person: Predicted

FIGURE 4.4

Predicted Per Capita GDP in the Production Model



Monotonicity in Capital

- As the first graph showed us, holding labor fixed, increasing capital should result in an increase in production.

- This means our production function will be **increasing** in capital.

- Mathematically, we can see this by

$$\frac{\partial F}{\partial K} = \alpha A K^{\alpha-1} L^{1-\alpha} > 0.$$

- Holding the number of workers constant, increasing the number of equipment, factories, machines, etc. implies an increase in production.

Marginal Product of Capital

- Notice that the slope of the graph became smaller over time.
- This means the output gained from one additional unit of capital (holding labor constant) became smaller as capital increased.
- The output gained from an additional unit of capital (holding labor constant) is called the **marginal product of capital**. It is the derivative of the production function with respect to capital.
- The key idea is if you have 10 workers and 5 machines, the increase in production you get from adding an additional machine is bigger than the increase you would get with 10 workers and 50 machines.

Marginal Product of Capital

- We established that the marginal product of capital is the derivative of production with respect to capital.
- The graph showed us that the marginal product of capital is **decreasing** in capital.
- Mathematically, we can see this from

$$\frac{\partial^2 F(K,L)}{\partial^2 K} = A\alpha(\alpha-1)K^{\alpha-2}L^{1-\alpha} < 0.$$

- The increase in production from an additional unit of capital (holding labor constant) get smaller with each additional unit of capital.

Predictions versus data

- We will now compare the models predictions versus that in the data.
- We will plug in capital per person and compare the output per person our model predicts versus the output per person in the data.
- In lecture 5 we will discuss how capital is measured and look at limitations and extensions. For now, think of it as the physical tools a country uses to produce output.
- Recall the parameters of the model are TFP $A = 1$ and the capital share $\alpha = 1/3$.

Predictions versus data

TABLE 4.3

The Model's Prediction for Per Capita GDP (U.S. = 1)

Country	Observed capital per person, $\bar{k}$	Predicted per capita GDP $y = \bar{k}^{1/3}$	Observed per capita GDP
United States	1.000	1.000	1.000
Switzerland	1.488	1.142	1.203
Japan	0.914	0.971	0.634
Italy	0.792	0.925	0.651
Spain	0.716	0.895	0.645
U.K.	0.714	0.894	0.707
Brazil	0.195	0.580	0.233
China	0.275	0.650	0.226
South Africa	0.165	0.548	0.200
India	0.082	0.434	0.107
Burundi	0.005	0.172	0.013

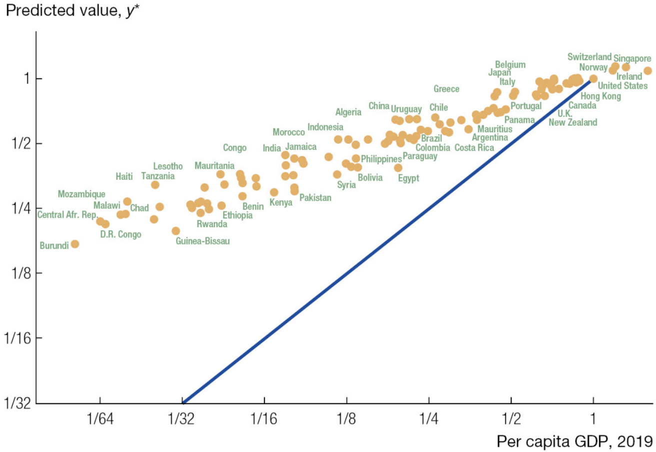
Predicted per capita GDP is computed as $\bar{k}^{1/3}$, that is, assuming no differences in productivity across countries. Data correspond to the year 2019 and are divided by the values for the United States.

Source: Penn World Table, Version 10.0.

Predictions versus data

FIGURE 4.5

The Model's Prediction for Per Capita GDP (U.S. = 1)



TFP

- The model repeatedly predicts countries should be richer than they actually are.
- In the model we assumed $A = 1$. That is, given the same capital labor ratio, each country should have the same output per person.
- Is it reasonable to expect the every country to be able to produce the same amount given the same resources?
  - Some countries have a more educated workforces than others and thus might be able to produce more with the same number of workers and capital.
  - Perhaps the quality of the machines, warehouses, etc. differs by country.
- To help our production function fit the data, we will relax the assumption $A = 1$.

TFP

- The central challenge of total factor productivity is that it is not directly measurable from the data.
- For capital per person, we can count the number of machines, factories, etc. in the economy and divide by the population.
- To allow TFP (A) to be numbers other than one, we will rearrange our production function to solve for A.
- It is important to note that what we are doing both assumes the model's accuracy to reality and enables us to fit the model exactly to the data.

TFP

- We will calculate total factor productivity so that the model fits exactly.
- For each country, we will calculate

$$
\begin{aligned}
y &= Ak^{\alpha} \\
A &= \frac{y}{k^{\alpha}}.
\end{aligned}
$$

  That is, we will divide each country's GDP per capita by capital per capita raised to $\alpha = 1/3$.

- Since we are not using independent measures of TFP (A) but instead computing it assuming the model, you can think of A as a measure of the gap between the modeled production function and reality.

TFP

TABLE 4.4

Measuring TFP So the Model Fits Exactly

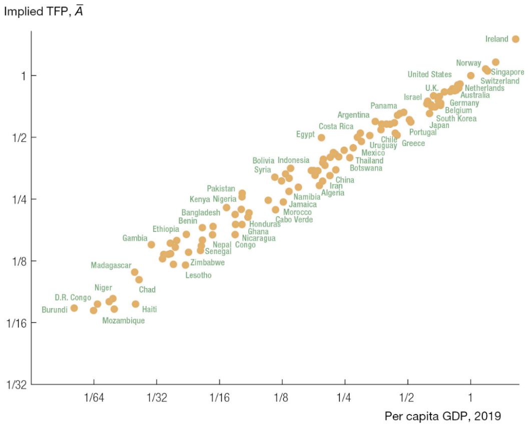
Country	Per capita GDP (y)	$\bar{k}^{1/3}$	Implied TFP ($\bar{A}$)
United States	1.000	1.000	1.000
Switzerland	1.203	1.142	1.054
U.K.	0.707	0.894	0.792
Japan	0.634	0.971	0.654
Italy	0.651	0.925	0.703
Spain	0.645	0.895	0.721
Brazil	0.233	0.580	0.402
South Africa	0.200	0.548	0.365
China	0.226	0.650	0.347
India	0.107	0.434	0.247
Burundi	0.013	0.172	0.074

Calculations are based on the equation $y = \bar{A}\bar{k}^{1/3}$. Implied productivity $\bar{A}$ is calculated from data on y and $\bar{k}$ for the year 2019, so that this equation holds exactly as $\bar{A} = y / \bar{k}^{1/3}$.

TFP

FIGURE 4.7

Measuring TFP So the Model Fits Exactly



TFP

- We saw above that the model over-predicted the level of output per person for countries with low capital per person levels.
- From our equation for A, the fact that poorer countries are not this wealthy implies they must have lower TFP levels.
- This leads us to three questions:
  1. Could we build a model where TFP is an endogenous variable? (Next lecture)
  2. What would other countries look like with U.S. TFP levels?
  3. Which contributes more towards differences in incomes across the world, differences in TFP or differences in capital?

TFP

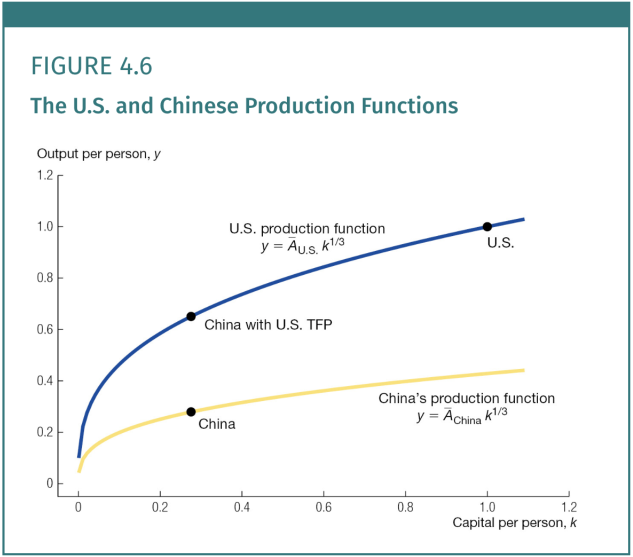


FIGURE 4.6
The U.S. and Chinese Production Functions
Output per person, y
1.2
1.0
0.8
0.6
0.4
0.2
0
U.S. production function
$y = \bar{A}_{U.S.} k^{1/3}$
U.S.
China with U.S. TFP
China's production function
$y = \bar{A}_{China} k^{1/3}$
China
0
0.2
0.4
0.6
0.8
1.0
1.2
Capital per person, k

Differences Across Countries

- Consider the five richest countries compared to the five poorest countries in 2019. This was approximately

$$\underbrace{\frac{y_{\text{rich}}}{y_{\text{poor}}}}_{75} = \underbrace{\frac{A_{\text{rich}}}{A_{\text{poor}}}}_{15} \cdot \underbrace{\left(\frac{k_{\text{rich}}}{k_{\text{poor}}}\right)^{\frac{1}{3}}}_{5}.$$

- Differences in TFP are roughly 3 times as important as differences in capital in explaining per capita GDP differences.

- This can be done between any two countries, usually TFP differences will explain $\approx 2/3$ and capital differences $\approx 1/3$.

- The richest countries are rich not only because they have more resources, but because they use them more efficiently.

Returns to Scale

- Suppose a country has a capital level K^* and labor level L^*. This means they'll have a production level $Y^* = F(K^*, L^*) = A(K^*)^\alpha (L^*)^{1-\alpha}$.
- If we double both the capital level and labor level, should we expect the output level to double? Should it expect the output level to more than double?
- The production function can account for this:

$$\begin{aligned} F(2K^*, 2L^*) &= A(2K^*)^\alpha (2L^*)^{1-\alpha} \\ &= 2^{\alpha+(1-\alpha)} A(K^*)^\alpha (L^*)^{1-\alpha} \\ &= 2Y^*. \end{aligned}$$

- We see that Y^* will exactly double. This argument works for any positive number. We say a production function that has this property exhibits **constant returns to scale**.

Returns to Scale

- Our production function exhibits constant returns to scale. Does this make sense?
- Suppose you own a factory that takes inputs and produces an output. If you find a suitable piece of land, build an additional factory, hire identical workers, and exactly double the inputs, you would expect your output to double. This reasoning is known as the **standard replication argument**.
- In your homework, you will show that

$$F(K, L) = AK^{\alpha_1} L^{\alpha_2}$$

is a CRS production function if $\alpha_1 + \alpha_2 = 1$. If $\alpha_1 + \alpha_2 > 1$ the production function exhibits increasing returns to scale. If $\alpha_1 + \alpha_2 < 1$ we see decreasing returns to scale.

Returns to Scale and Diminishing Returns

- It is worth emphasizing, though we have constant returns to scale we have diminishing returns to capital.
- If you double *both* capital and labor your output will double, if you double *ONLY* capital your output will less than double.
- The gains in production from an increase in capital get lower and lower the more capital a country has.

Increasing Returns to Scale

- Network effects: The value of the product or service increases as more people use it.
- Specialization and division of labor: Larger companies can finely divide tasks between workers, allowing them to specialize, reducing switching costs and increasing efficiency.
- Learning-by-doing: Repetition improves skills at the job and can make production more efficient over time.

Decreasing Returns

- Exhaustion of local resources: Expansion might require using less suitable resources. The best resources get used first, scaling requires using lower quality inputs.
- Resource congestion: Overuse of a shared or fixed input could reduce the production gains from increasing inputs.

Model

- Now that we've gained intuition with respect to the production function, we will define our model.
- For now, we will focus only on the firm side. This means we will think about the labor and capital demanded by the firm, taking the labor supply and capital supply as given.
- Later in the course, we will think about how labor and capital are supplied in an economy.

Model Setup: Demand

- The firm, taking the wage w and interest rate r as given, rents capital K and hires labor L to maximize profits, given by

$$\max_{K,L} AK^{\alpha}L^{1-\alpha} - rK - wL.$$

- Note that the price of the output is normalized to 1. So the wage w and interest rate r are expressed in terms of the output good.
- The good that is used to express prices is called the **numéraire**.

Model Setup: Demand

- We had

$$\max_{K,L} F(K,L) - rK - wL.$$

- To solve for the maximum, the firm will choose K and L such that the partial derivatives are zero.

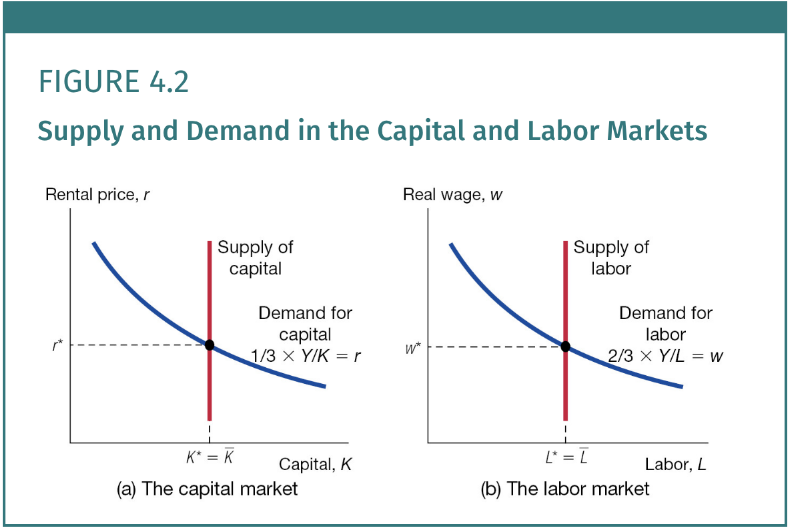
$$\begin{aligned} [K]: &\quad A\alpha K^{\alpha-1}L^{1-\alpha} - r = 0 \\ [L]: &\quad A(1-\alpha)K^{\alpha}L^{-\alpha} - w = 0. \end{aligned}$$

- This tells us that the interest rate r will be equal to the marginal product of capital and the wage w the marginal product of labor.

Model Setup: Supply

- The previous slide characterized the capital and labor **demanded** by the representative firm.
- Both capital and labor are **supplied** by a representative household.
- For now, we will take these as given and say capital K^* and labor L^* are inelastically supplied.

Model Setup: Supply



Model Setup: Labor Share

- The firm solved

$$\max_{K,L} AK^{\alpha}L^{1-\alpha} - rK - wL.$$

- Taking the derivative with respect to labor gives us

$$(1-\alpha)AK^{\alpha}L^{-\alpha} = w.$$

- We can use this to calculate the labor share

$$\begin{aligned}\frac{wL}{Y} &= \frac{wL}{AK^{\alpha}L^{1-\alpha}} \\ &= \frac{(1-\alpha)AK^{\alpha}L^{-\alpha}L}{AK^{\alpha}L^{1-\alpha}} \\ &= 1-\alpha.\end{aligned}$$

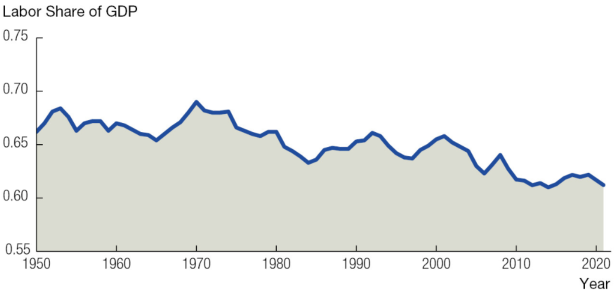
Model Setup: Labor Share

- We had
$$\frac{wL}{Y} = 1 - \alpha.$$
- $\frac{wL}{Y}$ represents the total sum paid to labor divided by the total GDP in an economy.
- Recall we had set $\alpha = 1/3$. This means the labor share should be 66%.

Model Setup: Labor Share

FIGURE 2.3

Labor's Share of GDP

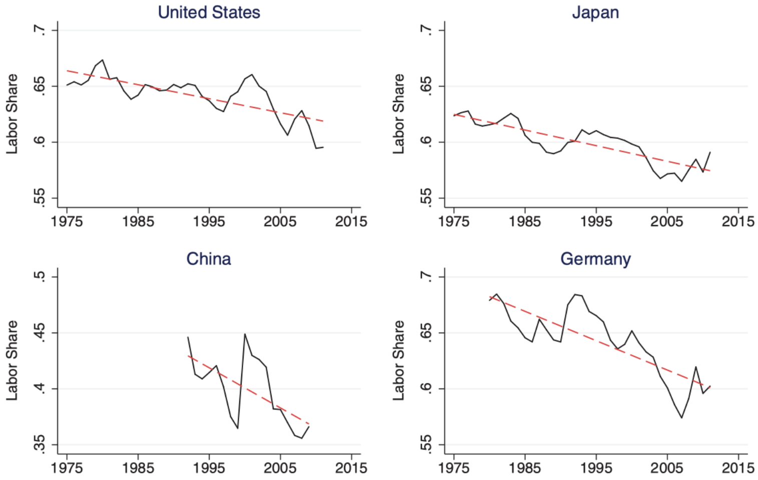


Source: U.S. Department of Commerce and author's calculations, Bureau of Economic Analysis, Table 1.10, bea.gov.

Labor share: Interesting trends

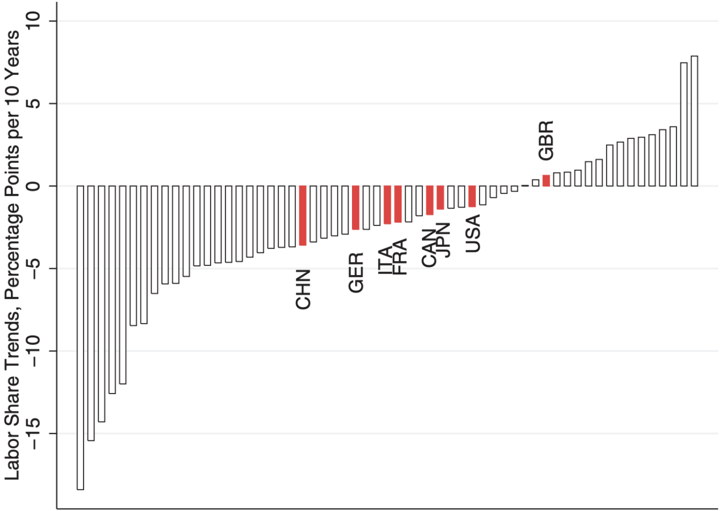
- Though the labor share hovered around 66%, you may have noticed a recent trend downwards.
- Trends in labor shares is an active area of research for economists.
- Your homework will have a short answer question in which you are tasked with coming up with a reason for the decline from a select group of papers.

Labor Share Trends[2]



[2]Karabarbounis and Neiman (2014) The Global Decline of the Labor Share

Labor Share Trends[3]



[3]Karabarbounis and Neiman (2014) The Global Decline of the Labor Share

Model Setup: Capital Share

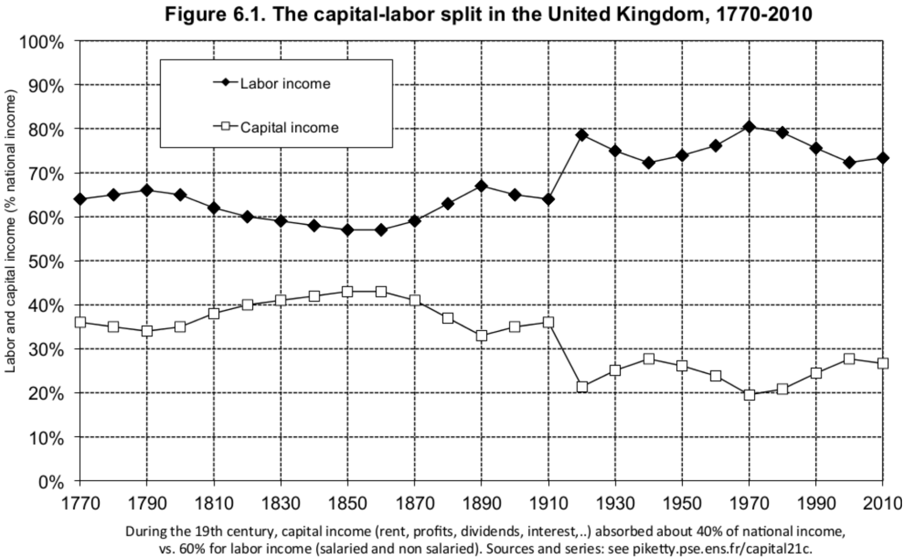
- We had the labor share $wL/Y = 1 - \alpha$.
- We can calculate the capital share using the same idea.
- Recall the profit maximization problem:

$$\max_{K,L} AK^{\alpha}L^{1-\alpha} - rK - wL.$$
$$\Rightarrow r = \alpha AK^{\alpha-1}L^{1-\alpha}.$$

Using this, we see that the capital share is given by

$$\frac{rK}{Y} = \frac{\alpha AK^{\alpha-1}L^{1-\alpha}K}{AK^{\alpha}L^{1-\alpha}}$$
$$= \alpha.$$

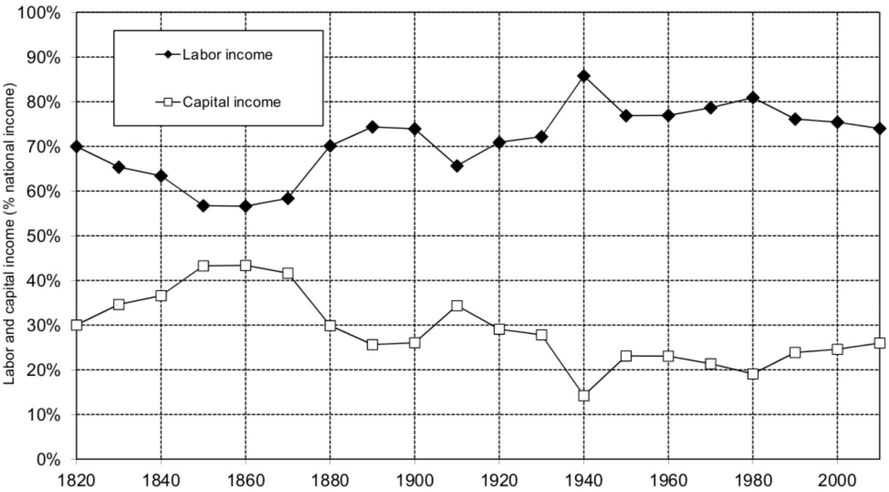
Model Setup: Capital share over time[4]



[4] Piketty (2014) Capital in the 21st century

Model Setup: Capital share over time[5]

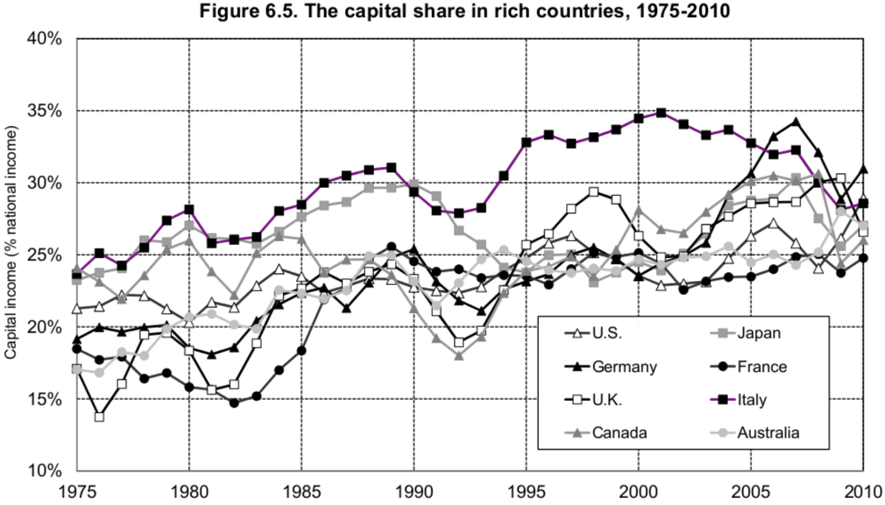
Figure 6.2. The capital-labor split in France, 1820-2010



In the 21st century, capital income (rent, profits, dividends, interest,..) absorbs about 30% of national income, vs. 70% for labor income (salaried and non salaried). Sources and series: see piketty.pse.ens.fr/capital21c.

[5] Piketty (2014) Capital in the 21st century

Capital Share over time[6]



Capital income absorbs between 15% and 25% of national income in rich countries in 1970, and between 25% and 30% in 2000-2010. Sources and series: see piketty.pse.ens.fr/capital21c

[6] Piketty (2014) Capital in the 21st century

Model Setup: Profits

- We had

$$\frac{wL}{Y} + \frac{rK}{Y} = 1 - \alpha + \alpha = 1.$$

- Output can be partitioned between workers and owners of capital.
- We do not have to worry about owners of the firms because with a CRS production function and marginal costs, firms will make zero profits.

Model Setup: Profits

- For intuition on how our model implies zero profits, suppose that there existed capital level K^* and labor L^* such that the firm made profit $\pi > 0$, or

$$\pi = A\,(K^*)^{\alpha}\,(L^*)^{1-\alpha} - rK^* - wL^* > 0.$$

  Since we have a CRS production function, note that if the firm doubles its inputs then

$$\begin{aligned} A\,(2K^*)^{\alpha}\,(2L^*)^{1-\alpha} - r2K^* - w2L^* &= 2(A\,(K^*)^{\alpha}\,(L^*)^{1-\alpha} - rK^* - wL^*) \\ &= 2\pi > \pi. \end{aligned}$$

  The firm can double its profit by doubling its input. This same logic works for any number (double, triple, quadruple, x1,000,000, etc.)

- The firm is solving a maximization problem with no maximum, which isn't well-defined.

Model Setup: Profits

- The previous slide explained how we know our firm problem won't yield an answer with positive profit.
- The intuition for why our firm problem won't yield an answer with negative profit is much simpler.
- The firm can choose capital K and labor L. Note that the *only* costs they face are marginal costs, not fixed cost. If they employ 0 capital and 0 labor then they face zero costs.
- So we know

$$0 \leq \max_{K,N} AK^{\alpha}L^{1-\alpha} - RK - WL \leq 0$$

which means the our model will yield zero profits.

Model

- Output is produced using a Cobb-Douglas technology

$$Y = F(K, L) = AK^{\alpha} L^{1-\alpha}$$

- Optimal rule for hiring capital

$$r = \alpha \frac{Y}{K}.$$

- Optimal rule for hiring labor

$$w = (1 - \alpha) \frac{Y}{K}.$$

- Supply of capital from households, fixed $\bar{K}$, equals demand of capital $K = \bar{K}$.
- Supply of labor from households, fixed $\bar{L}$, equals demand of labor $L = \bar{L}$.

Model

- The solution involves the firm choosing capital K and labor L to maximize its profits.
- Firms keep hiring until the marginal product of capital equals the interest rate and the marginal product of labor equals the wage.
- The solution also ensure the labor and capital demanded from this maximization problem equals the labor and capital supplied. In economic models, the condition in which supply equals demand is called the market clearing condition.

Questions

- Suppose that
$$r > \alpha A K^{\alpha-1} L^{1-\alpha}.$$
That is, the interest rate is greater than the marginal product of capital. Should the firm increase or decrease its demand for capital?
- Suppose
$$w < (1-\alpha) A K^{\alpha} L^{-\alpha}.$$
That is, the wage is less than the marginal product of labor. Should the firm increase or decrease its demand for labor?

Substitutes versus Complements

- Recall from microeconomics the idea of substitutes and complements.
- Substitutes: inputs that can replace one another for production. Two inputs are substitutes if an increase in the use of one can compensate for using less of the other, while keeping output constant.
- Complements: inputs that are used together in production, where the productivity of one increases with the use of the other. If one input is removed, the effectiveness of the other diminishes.
- Suppose a firm receives new capital. With our production function, is it better to hire more workers or use the capital to replace the workers?

Substitutes or Complements?

- We want to test whether capital and labor are complements or substitutes.
- We can calculate this by seeing if the marginal product of capital is increasing in labor (or vice-versa)

$$\frac{\partial^2 F}{\partial L \partial K} = A\alpha(1-\alpha)AK^{\alpha-1}L^{-\alpha} > 0.$$

- Capital and labor are **complements** in our model.

Wrapping Up

- We constructed a model of production that takes in two measureable inputs, capital and labor, and gives us the output.
- We used this to test our model against real-world data by looking at output per person and capital per person.
- We found that TFP was relatively more important in predicting income differences between countries than capital per person.
- Question for the next few lectures: How can we better understand why TFP differs so much across countries?