Solow Model[1]

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June 2025

[1] All graphs are from our textbook unless said otherwise.

Review

- Previously, we explored the production function

$$F(K, L) = AK^{\alpha}L^{1-\alpha}$$

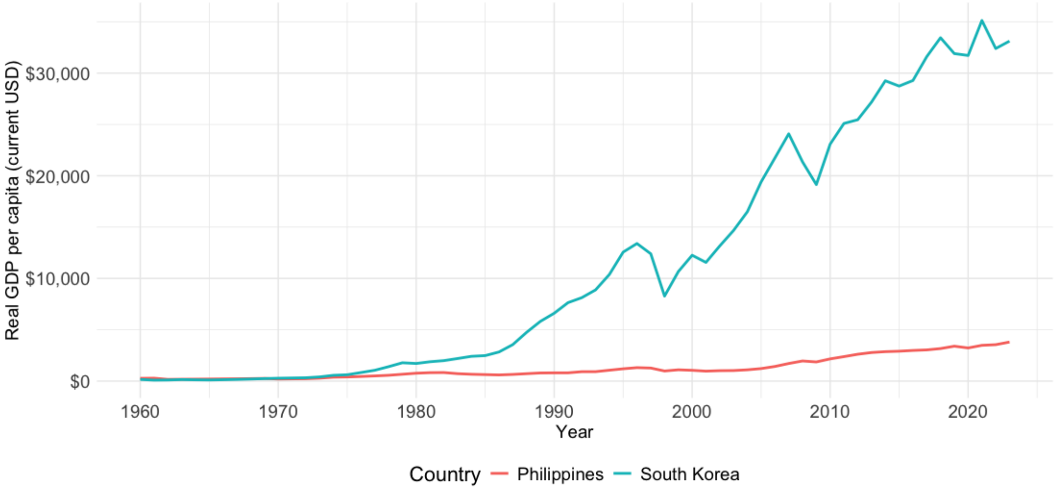
  and its properties.
- We discussed the importance of both capital and TFP in determining output per person levels across countries.
- While interesting, there was no concept of growth over time in this framework.

Solow: Motivation[2]

- In 1960 South Korea and the Philippines were relatively similar.
    1. Both had per capita GDP of around \$1,500 (around 10% of U.S. level).
    2. Both had similar populations (25 million in Korea and 28 million in the Philippines)
    3. Both had similar shares of the population living near the capital city (27% near Manila and 28% near Seoul).
    4. Both had similar populations of working age adults (slightly over half).
    5. 5 % of Koreans were in college versus 13% in the Philippines.
- By 2019, per capita GDP in Korea reached \$42,000 while the Philippines was only \$8,500.

[2] See Robert E. Lucas Jr., "Making a Miracle", Econometrica (1993)

Solow: Motivation[3]



[3]World Bank via FRED

Solow: Motivation

- South Korea and the Philippines were in relatively similar positions in 1960.
- Today, South Korean GDP per capita is around 5 times higher than the Philippines.
- Why did South Korea grow so much faster than the Philippines?
- The Solow model gives us a theory for the determinants of growth over time.

The Solow Model

- Let $t = 1, 2, 3, ...$ denote sequential periods of time.
- In each period, output will work the same as previously

$$Y_t = F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}.$$

- For simplicity, we will assume $L_t = \bar{L}$ each time period. Our focus will be on capital.

The Solow Model

- The Solow Model uses the production function from last lecture, adding **capital accumulation**.
- In each period, a fraction $\bar{s}$ of output Y_t is **invested** and used as capital in production next period.
  - Note that $\bar{s}$ is constant each period, it doesn't vary with time.
- In each period, a fraction δK_t of capital depreciates.
- Capital accumulates by
$$K_{t+1} = \bar{s}Y_t + (1-\delta)K_t.$$
- Next period's capital equals your previous capital after depreciation plus investments.

The Solow Model

- Putting capital accumulation with the production function, we have

$$Y_t = F(K_t, \bar{L})$$
$$K_{t+1} = \bar{s}Y_t + (1-\delta)K_t.$$

- Capital that isn't invested $(1-\bar{s})Y_t$ is consumed C_t. This means

$$C_t + \bar{s}Y_t = Y_t.$$

Solow: Mechanics

1. In $t = 0$, begin with labor supply $\bar{L}$ and initial capital K_0.
2. Using initial capital K_0 and labor $\bar{L}$, get $Y_0 = F(K_0, \bar{L})$. Then use period 0 output Y_0 and capital K_0 to obtain
$$K_1 = \bar{s}Y_0 + (1 - \delta)K_0.$$
3. Using period 1 capital K_1 and labor $\bar{L}$, get $Y_1 = F(K_1, \bar{L})$. Then use period 1 output Y_1 and capital K_1 to get
$$K_2 = \bar{s}Y_1 + (1 - \delta)K_1.$$
4. Repeat this process for an arbitrary number of times to get a sequence of capital and output.

- How do we analyze this model?

Solow: Analysis

- We want to find a capital level K_{ss} and output Y_{ss} such that

$$K_{ss} = \bar{s}Y_{ss} + (1 - \delta)K_{ss}.$$

- The values K_{ss}, Y_{ss} are called the **steady-state** level of capital and output. They don't change over time.
- We can see how these values change for different values of the investment rate $\bar{s}$ and depreciation levels δ.

Solow: Solving

- In our graph of South Korea and the Philippines, we cared about output per person $y_t \equiv Y_t/\bar{L}$.
- Let $k_t \equiv K_t/\bar{L}$ and $y_t \equiv \frac{Y_t}{\bar{L}}$.
- We know from the previous lecture that we can turn

$$Y_t = AK_t^{\alpha}\bar{L}^{1-\alpha}$$

into

$$y_t = Ak_t^{\alpha}$$

Solow: Solving

▷ We also have the capital accumulation equation

$$
\begin{aligned}
K_{t+1} &= \bar{s}Y_t + (1-\delta)K_t \\
\frac{K_{t+1}}{\bar{L}} &= \bar{s}\frac{Y_t}{\bar{L}} + (1-\delta)\frac{K_t}{\bar{L}} \\
k_{t+1} &= \bar{s}y_t + (1-\delta)k_t.
\end{aligned}
$$

▷ The steady state we want is then

$$k_{ss} = \bar{s}y_{ss} + (1-\delta)k_{ss}.$$

▷ We now have the production function and capital accumulation equation written in terms of output per person and capital per person.

Solow: Steady State

- We had the equation for a steady state

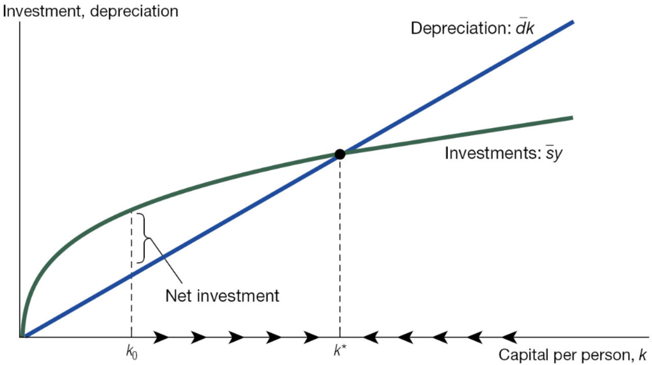
$$\begin{aligned} k_{ss} &= \bar{s}y_{ss} + (1-\delta)k_{ss} \\ k_{ss} - (1-\delta)k_{ss} &= \bar{s}y_{ss} \\ \delta k_{ss} &= \bar{s}y_{ss}. \end{aligned}$$

- A steady state occurs when investments $\bar{s}y_{ss}$ exactly offset depreciation δk_{ss}.

Solow: Steady State

FIGURE 5.1

The Solow Diagram



Steady State Guarantee

- Why are we guaranteed a steady state?
- Note in the previous graph that the investment function $\bar{s}y$ is concave in capital per person.
- This happens because we have diminishing returns to capital per person since $\bar{s}y = \bar{s}k^{\alpha}$.
- Because of diminishing returns to capital per person, the slope of the production per person function is decreasing in capital, so inevitably the depreciation δk (which has a constant slope δ) will intersect it.

Solow: Steady State

- We had the equation for a steady state as

$$\bar{s}y_{ss} = \delta k_{ss}.$$

- Using $y_{ss} = Ak_{ss}^{\alpha}$, we can solve

$$\bar{s}Ak_{ss}^{\alpha} = \delta k_{ss}$$

  and get

$$k_{ss} = \left(\frac{\bar{s}A}{\delta}\right)^{\frac{1}{1-\alpha}}.$$

- Plugging this into our output equation $y_{ss} = Ak_{ss}^{\alpha}$, we get

$$y_{ss} = A^{\frac{1}{1-\alpha}}\left(\frac{\bar{s}}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

Solow: Steady State

- We had

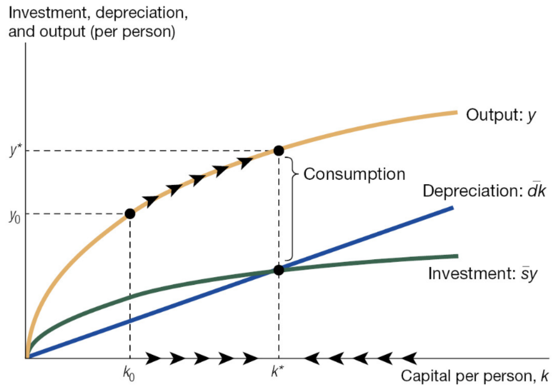
$$k_{ss} = \left(\frac{\bar{s}A}{\delta}\right)^{\frac{1}{1-\alpha}}$$
$$y_{ss} = A^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}}{\delta}\right)^{\frac{\alpha}{1-\alpha}} .$$

- The Solow model has an **analytical solution**. This means we can write down a solution formula using **only** terms given in the problem.
- Analytical solutions are very useful in models, as they allow us to consider how the optimal solution changes if we alter parameters.

Solow: Steady State

FIGURE 5.2

The Solow Diagram with Output



Analytical Solutions

- Notice again how

$$k_{ss} = \left(\frac{\bar{s}A}{\delta}\right)^{\frac{1}{1-\alpha}}$$
$$y_{ss} = A^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

  only depend on parameters. That is, our solution to y_{ss} does include k_{ss}.

- In this class, if you are asked to solve a model, your answer should depend *only* on parameters.
- For the rest of the course, you will often solve a model and economically interpret the answer you get.

Solow: Comparative Statics

- We had

$$y_{ss} = A^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

- Suppose we increase depreciation δ, how does that change steady state output per person y_{ss}?
- Taking a derivative of output per person (y_{ss}) with respect to δ, we have

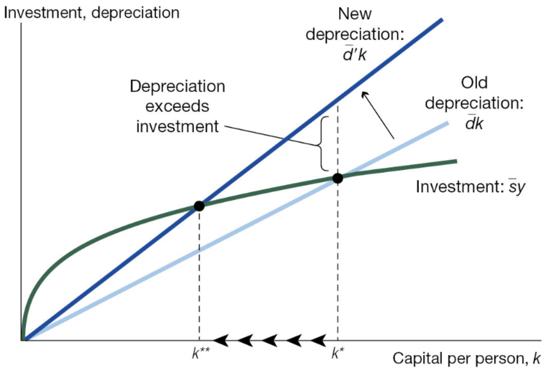
$$\frac{\partial y_{ss}}{\partial \delta} = -\frac{\alpha}{1-\alpha} \left(\frac{y_{ss}}{\delta}\right) < 0.$$

- This tells us that equilibrium output per person (y_{ss}) **decreases** as we increase depreciation δ.

Solow: Comparative Statics

FIGURE 5.6

A Rise in the Depreciation Rate



Solow: Comparative Statics

- Suppose we increase the savings rate $\bar{s}$, how does this change steady state output per person (y_{ss})?
- Taking a derivative of output per person with respect to $\bar{s}$ we have

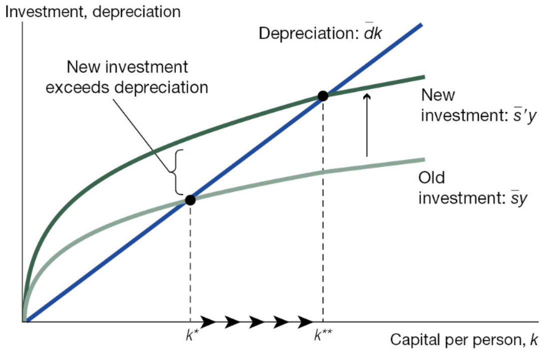
$$\frac{\partial y_{ss}}{\partial \bar{s}} = \frac{\alpha}{1-\alpha}\left(\frac{y_{ss}}{\bar{s}}\right) > 0.$$

- This tells us that equilibrium output per person (y_{ss}) **increases** as we increase savings rate $(\bar{s})$.

Solow: Comparative Statics

FIGURE 5.4

An Increase in the Investment Rate



Solow: Comparative Statics

- We've seen that **increasing** the depreciation rate (δ) **decreases** the steady-state level of output per person (y_{ss}).
- We've seen that **increasing** the savings rate $(\bar{s})$ **increases** the steady state level of output per person (y_{ss}).
- In the last lecture, we saw how poorer countries had lower TFP levels. In the Solow model, we can see that an increase in TFP (A) increases y_{ss}.

$$\frac{\partial y_{ss}}{\partial A} = \left(\frac{\alpha}{1-\alpha}\right) A^{\frac{2\alpha-1}{1-\alpha}} \left(\frac{\bar{s}}{\delta}\right)^{\frac{\alpha}{1-\alpha}} > 0.$$

- The Solow model supports our observation that richer countries use resources more effectively.

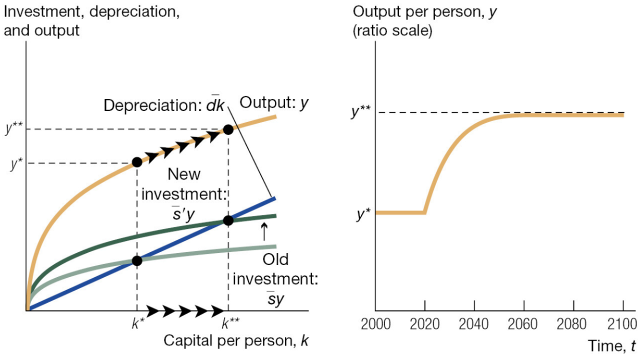
Solow: Transition Path

- So far we have looked at differences in steady states of output per person using different parameters.
- What happens if we are in a steady state, and one of our parameters change?
- We can calculate the new steady state, but what will the path to that new steady state look like?

Solow: Transition Path

FIGURE 5.5

The Behavior of Output after an Increase in s



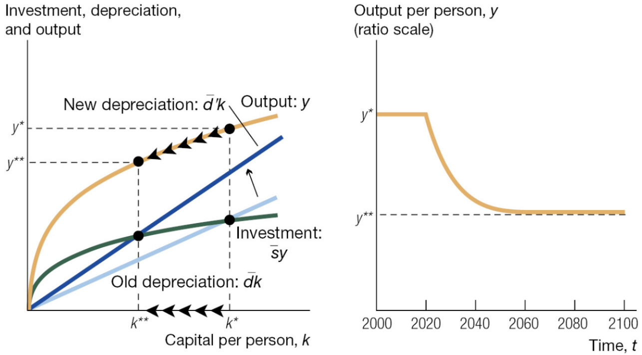
(a) The Solow diagram with output.

(b) Output per person over time.

Solow: Transition Path

FIGURE 5.7

The Behavior of Output after an Increase in $\bar{d}$



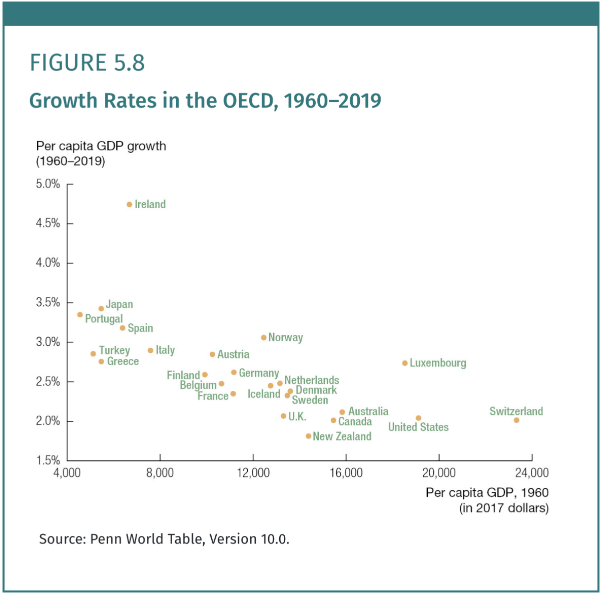
(a) The Solow diagram with output.

(b) Output per person over time.

Solow: Transition Path

- The Solow model implies that the further beneath a country is from its steady state, the faster it will grow.
  - If two countries have the same output per person today but one is 95% of its steady state while the other is 50%, the second country will have a higher output per person tomorrow.
- If every country had the same TFP (A), investment rate $(\bar{s})$, and depreciation (δ), we would expect poorer countries to grow **faster** than richer countries.

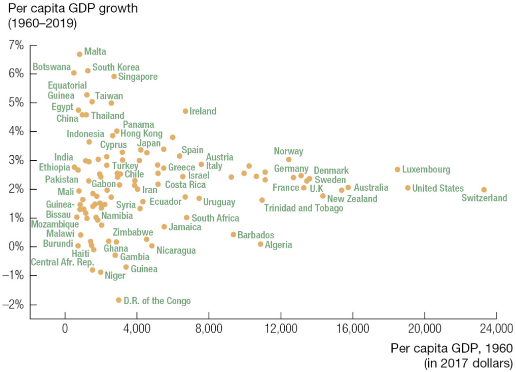
Solow: To The Data



Solow: To The Data

FIGURE 5.9

Growth Rates around the World, 1960–2019



Source: Penn World Table, Version 10.0.

Solow: To the Data

- Recall that in a steady state we have

$$\bar{s}y_{ss} = \delta k_{ss}$$

  which implies

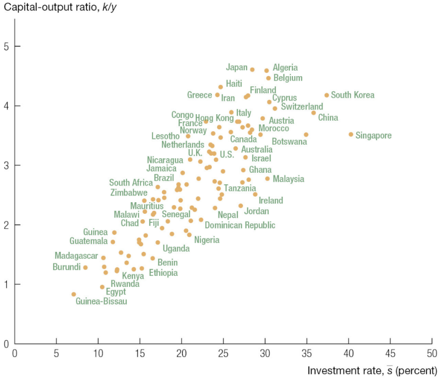
$$\frac{k_{ss}}{y_{ss}} = \frac{\bar{s}}{\delta}.$$

- This equation tells us the capital-output ratio k_{ss}/y_{ss} must equal the investment rate divided by depreciation.
- Countries vary greatly in investment rates $\bar{s}$, while depreciation rates δ are more steady country to country.
- This means we should expect higher investment rate $\bar{s}$ to imply a higher capital output ratio.

Solow: To the Data

FIGURE 5.3

Explaining Capital in the Solow Model



Source: Penn World Table, Version 10.0. The capital-output ratio is measured in the year 2019, while the investment rate is averaged over the period 1990 to 2019.

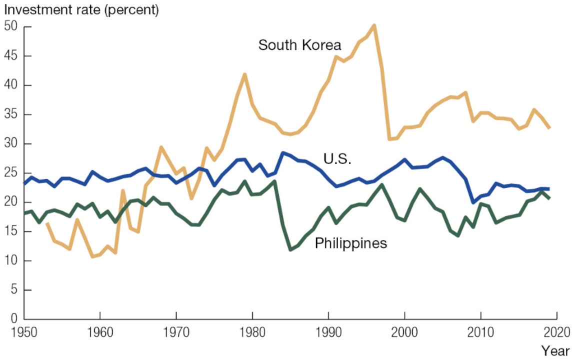
Solow: To the Data

- Our initial motivation was the massive difference in the economic trajectories of South Korea and the Philippines.
- The Solow model tells us that the further beneath a country is from its steady state, the fast it will grow.
- If the investment rate of South Korea was higher than the Philippines, we would expect it to growth faster.

Solow: To the Data

FIGURE 5.10

Investment in South Korea and the Philippines, 1950–2019



Source: Penn World Table, Version 10.0.

Solow: Benefits

- The Solow model takes the production function and adds capital accumulation.
- It provides a theory that explains what determines a country's output per person in the long run.
- It explains why countries differ in their growth rates across the world.
  - The further a country is below its steady state value, the faster it grows.

Solow: Population Growth

- So far we've assumed the population remained constant $L_t = \bar{L}$.
- We now relax this assumption. Instead, assume the population grows at a constant rate $\bar{n}$.
- That is,

$$\frac{\Delta L_t}{L_t} = \bar{n}.$$

- Note that we can still transform our output equation

$$\begin{aligned}
\frac{Y_t}{L_t} &= \frac{AK_t^{\alpha} L_t^{1-\alpha}}{L_t} \\
&= A\left(\frac{K_t}{L_t}\right)^{\alpha} \\
&= Ak_t^{\alpha}.
\end{aligned}$$

Solow: Population Growth

- When we divide

$$K_{t+1} = \bar{s}Y_t + (1-\delta)K_t$$
$$\frac{K_{t+1}}{L_t} = \bar{s}\frac{Y_t}{L_t} + (1-\delta)\frac{K_t}{L_t}$$

  we can no longer obtain our equation $k_{t+1} = \bar{s}y_t + (1-\delta)k_t$.

- Instead, we can use the fact that the growth rate of a ratio (k_t) is approximately the difference between the two growth rates

$$\frac{\Delta k_{t+1}}{k_t} = \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta L_{t+1}}{L_t}.$$

Solow: Population Growth

- We had

$$\begin{aligned}\frac{\Delta k_{t+1}}{k_t} &= \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta L_{t+1}}{L_t} \\ &= \frac{K_{t+1} - K_t}{K_t} - \bar{n} \\ &= \frac{\bar{s}Y_t - \delta K_t}{K_t} - \bar{n} \\ &= \bar{s}\frac{Y_t}{K_t} - (\bar{n} + \delta) \\ &= \bar{s}\frac{y_t}{k_t} - (\bar{n} + \delta).\end{aligned}$$

So

$$\Delta k_{t+1} = \bar{s}y_t - (\bar{n} + \delta)k_t.$$

Solow: Population Growth

- We now have two equations

$$\begin{aligned} y_t &= Ak_t^\alpha \\ \Delta k_{t+1} &= \bar{s}y_t - (\bar{n}+\delta)k_t. \end{aligned}$$

- In a steady state, we'll have

$$\begin{aligned} y_{ss} &= Ak_{ss}^\alpha \\ 0 &= \bar{s}y_{ss} - (\bar{n}+\delta)k_{ss}. \end{aligned}$$

- In a steady state the capital stock is not changing.

Solow: Population Growth

- We had two equations for our steady state and two unknowns

$$\begin{aligned} y_{ss} &= Ak_{ss}^{\alpha} \\ 0 &= \bar{s}y_{ss} - (\bar{n} + \delta)k_{ss}. \end{aligned}$$

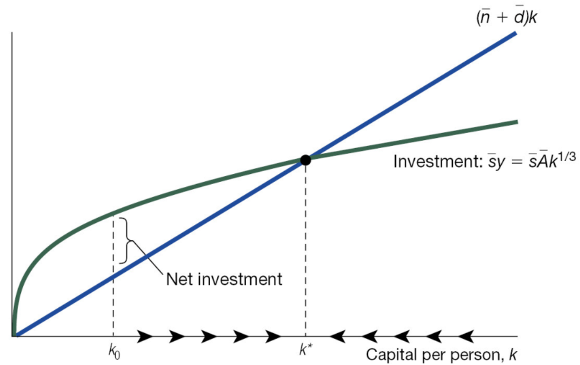
- We can solve for the steady state for capital and output to get

$$\begin{aligned} k_{ss} &= \left(\frac{\bar{s}A}{\bar{n}+\delta}\right)^{\frac{1}{1-\alpha}} \\ y_{ss} &= A^{\frac{1}{1-\alpha}}\left(\frac{\bar{s}}{\bar{n}+\delta}\right)^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

Solow: Population Growth

FIGURE 5.11

The Solow Diagram with Population Growth



Solow: Population Growth

- In your homework, you will see the relationships between the investment rate $(\bar{s})$, depreciation (δ), and TFP (A) with steady state output per person (y_{ss}) hold.
- What do you expect the relationship between steady state output and the population growth rate to be?

Solow: Population Growth

- We had
$$y_{ss} = A^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}}{\bar{n} + \delta} \right)^{\frac{\alpha}{1-\alpha}}.$$
- We can calculate
$$\frac{\partial y_{ss}}{\partial \bar{n}} = -\frac{\alpha}{1-\alpha} \left(\frac{y_{ss}}{\bar{n} + \delta} \right) < 0.$$
- More people are being born who need k_{ss} units of capital, which means some of the capital must go towards this purpose rather than towards increasing capital per person.

Solow: Population Growth

- Adding population growth now adds the feature of long-run economic growth to the Solow model.
- While the output per person $(y_{ss)}$ is constant, total output $Y_t = L_t y_{ss}$ grows at rate $\bar{n}$.
- With or without population growth, the Solow model **does not** feature growth in the long run of output per person
- Without population growth, total output does not have growth in the long run of output. However, with a constant population growth rate, the Solow model features long run growth of output.

Solow: Limitations

- In the first lecture we saw many graphs showing us that output per person had been growing consistency across the world over the past several decades.
- We previously showed that differences in investments in physical capital explain only a small part of the differences in output per person across countries.
- The Solow model does not explain **why** different countries have different productivities A or investment rates $\bar{s}$.
  - **Why** is it that South Korea's investment rate increased in the 1960's?

Moving Forward: Questions to Ask

- So far, TFP (A) has been a given constant. How would we create a model where TFP changes over time?
- How could we create a model that gives us long-run economic growth in output per person?