Human Capital

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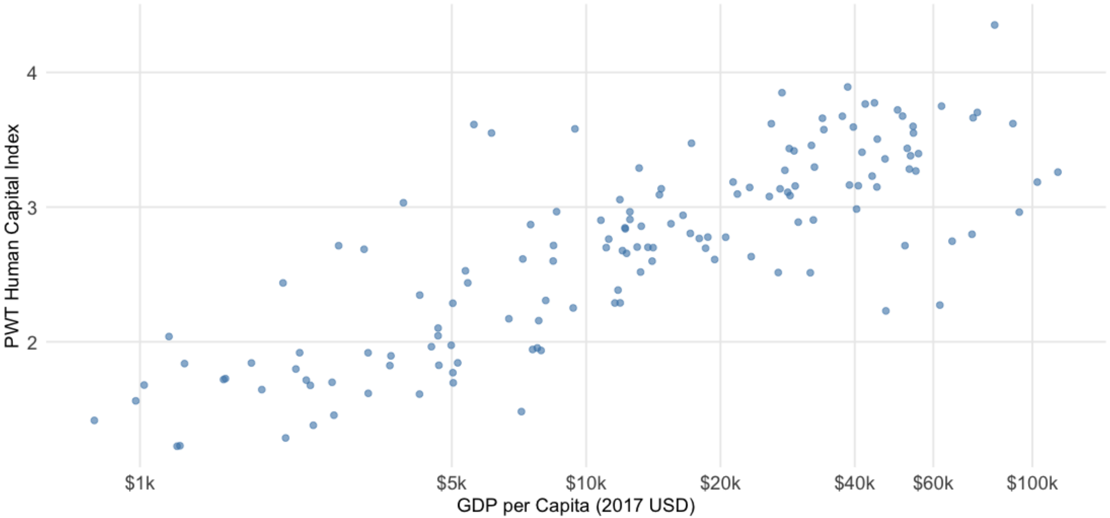
June 2025

Review: Last Lecture[1]



[1] Source: Penn World Tables v. 10.01

Review: Last Lecture

- We saw last lecture that indexes based on years of schooling and returns to education are heavily correlated with gdp per capita.
- This is true in general, richer countries tend to have more educated workforces.
- To what extent can human capital help explain differences in cross-country economic outcomes?

Education Differences

- To test this, we will consider a Solow model with human capital.
- We will replace labor with human capital.
- We will ignore TFP differences.
- These alterations allow us to see the level of differences between countries explained by only physical and human capital.

Basic Structure

- Consider the following production function

$$Y_t = K_t^{\alpha}[(1-s_H)H_t]^{1-\alpha}$$

where

$$\begin{aligned} K_t &: \text{ Physical capital stock} \\ H_t &: \text{ Human capital stock} \\ 1-s_H &: \text{ Share of time allocated to market production.} \end{aligned}$$

- Individuals allocate fraction $0 < s_H < 1$ of their time to enhancing human capital and $1 - s_H$ to market production.

Physical Capital Accumulation

- Physical capital accumulates identically to the Solow model.
- We have
$$K_{t+1} = s_K Y_t + (1-\delta)K_t$$
- In the Solow model we had output per person and capital per person.
- Now, we will define
$$y_t \equiv \frac{Y_t}{H_t} : \quad \text{Output per effective human capital}$$
$$k_t \equiv \frac{K_t}{H_t} : \quad \text{Capital per effective human capital}$$

Human Capital Accumulation

- Recall $1 - s_H$ was the share in market production.
- s_H is the share of time in enhancing human capital.
- Human capital accumulates according to

$$H_{t+1} = (s_H H_t)^\sigma + (1-\delta) H_t$$

  where

  σ : Governs the intensity of human capital in education production.

- Human capital depreciates at rate δ.

Explaining σ

- We had

$$H_{t+1} = (s_H H_t)^\sigma + (1-\delta)H_t.$$

- We can interpret σ as representing the efficiency of education technology in a country.
- A country with a high quality education system can produce more human capital given the same investment $s_H H_t$ compared to a country whose education system is riddled with attrition, low attendance, and poor quality of instruction.
- We will set $0 < \sigma < 1$.

Steady State

- Recall that when analyzing the Solow model we looked for a steady state.
- We will do the same here, that is, we will search for when $H_{t+1} = H_t$ and $K_{t+1} = K_t$. For our equation for human capital, we will have

$$\begin{aligned}
H_{t+1} &= (s_H H_t)^\sigma + (1-\delta)H_t \\
H_{ss} &= (s_H H_{ss})^\sigma + (1-\delta)H_{ss} \\
\delta H_{ss} &= (s_H H_{ss})^\sigma \\
\delta H_{ss}^{1-\sigma} &= s_H^\sigma \\
H_{ss} &= \left(\frac{s_H^\sigma}{\delta}\right)^{\frac{1}{1-\sigma}} .
\end{aligned}$$

Elasticity

- In previous lectures, we took derivatives of model solutions with respect to parameters to see the effect of altering the parameter on the model solution.
- In this lecture, we will take our characterization of the impact of a parameter on an endogenous variable one step further.
- We will quantify the responsiveness of the steady state human capital level H_{ss} with respect to depreciation δ and the share of time enhancing human capital (s_H) by introducing and calculating elasticities.

Elasticity

- Suppose with depreciation δ we get steady state human capital level H_{ss} and with $\delta' > \delta$ we get steady state human capital level $\delta' > \delta$.

- The formula for the elasticity of steady state human capital with respect to depreciation (δ) is given by

$$\begin{aligned}
\varepsilon_{H_{ss},\delta} &= \frac{\frac{H_{ss}^{'}-H_{ss}}{H_{ss}}}{\frac{\delta'-\delta}{\delta}} \\
&= \frac{H_{ss}^{'} - H_{ss}}{\delta' - \delta}\left(\frac{\delta}{H_{ss}}\right) \\
&= \frac{\partial H_{ss}}{\partial \delta}\left(\frac{\delta}{H_{ss}}\right).
\end{aligned}$$

- We can calculate the elasticity of steady state human capital with respect to δ by taking the partial derivative of H_{ss} with respect to δ and multiplying by the fraction $\frac{\delta}{H_{ss}}$.

Elasticity

- We formulated a tool we could use to measure the change of variables of interest from altering parameters.
- It can be shown that

$$\varepsilon_{H_{ss},\delta} = \frac{\partial H_{ss}}{\partial \delta}\left(\frac{\delta}{H_{ss}}\right) = \frac{\partial \ln H_{ss}}{\partial \ln \delta}.$$

- That is, the elasticity of steady state human capital (H_{ss}) with respect to depreciation (δ) is simply the derivative of the log of steady state human capital with respect to the log of depreciation.

Elasticity

- We had both

$$H_{ss} = \left(\frac{s_H^\sigma}{\delta}\right)^{\frac{1}{1-\sigma}}$$

$$\varepsilon_{H_{ss},\delta} = \frac{\partial \ln H_{ss}}{\partial \ln \delta}.$$

- Noting that

$$\ln(H_{ss}) = \frac{\sigma}{1-\sigma}\ln(s_H) - \frac{1}{1-\sigma}\ln(\delta),$$

we get

$$\varepsilon_{H_{ss},\delta} = -\frac{1}{1-\sigma}.$$

Elasticity

- We found that
$$\varepsilon_{H_{ss},\delta} = -\frac{1}{1-\sigma}.$$
- Since $-\frac{1}{1-\sigma} < 0$, we know that increasing the depreciation rate (δ) decreases the steady state level of human capital (H_{ss}). This we could have seen by just taking $\frac{\partial H_{ss}}{\partial \delta}$.
- A 1% increase in the depreciation rate (δ) will *decrease* steady state human capital (H_{ss}) by $\frac{1}{1-\sigma}\%$.
- Note that since $0 < \sigma < 1$, then $\frac{1}{1-\sigma} > 1$.

Elasticity

- Elasticities allow us to compare the magnitudes of changes from different variables.
- Recall that we had

$$\ln(H_{ss}) = \frac{\sigma}{1-\sigma}\ln(s_H) - \frac{1}{1-\sigma}\ln(\delta),$$

- We know from above that if we want to calculate the elasticity of steady state human capital (H_{ss}) with respect to the share of time enhancing human capital production then we'd compute

$$\begin{aligned}\varepsilon_{H_{ss},s_H} &= \frac{\partial \ln H_{ss}}{\partial \ln s_H} \\ &= \frac{\sigma}{1-\sigma}.\end{aligned}$$

Elasticity

- We had
$$\varepsilon_{H_{ss},s_H} = \frac{\sigma}{1-\sigma}.$$
- Since $\varepsilon_{H_{ss},s_H} > 0$, increasing individuals share of time in the human capital production sector (s_H) will increase steady state human capital (H_{ss}).
- Note that doing this might reduce steady state output, but output does **not** factor into human capital in this model.
- A 1% increase in the share of time producing human capital increases steady state human capital by $\frac{\sigma}{1-\sigma}\%$.

Elasticity

- We had

$$\varepsilon_{H_{ss},\delta} = -\frac{1}{1-\sigma}$$
$$\varepsilon_{H_{ss},s_H} = \frac{\sigma}{1-\sigma}.$$

- Notice that since $0 < \sigma < 1$, we have

$$\left|\varepsilon_{H_{ss},s_H}\right| < \left|\varepsilon_{H_{ss},\delta}\right|.$$

- So a 1% change in the depreciation rate has a larger effect in magnitude than a 1% change in the share of time in human capital.
- According to the model, a policy that prolongs the life of human capital by slowing depreciation 1% increases steady state human capital more than a program that increases time enhancing human capital (s_H) 1%.

Steady State Physical Capital

- We can rewrite our capital accumulation equation in the steady state to be in terms of per human capital

$$
\begin{aligned}
K_{t+1} &= s_K Y_t + (1-\delta) K_t \\
K_{ss} &= s_K Y_{ss} + (1-\delta) K_{ss} \\
\delta K_{ss} &= s_K Y_{ss}
\end{aligned}
$$

- This is the same condition for a steady state that we had in the original Solow model. Investment in capital must equal depreciation.

- We can divide both side by steady state human capital H_{ss} to get

$$
\begin{aligned}
\delta \frac{K_{ss}}{H_{ss}} &= s_K \frac{Y_{ss}}{H_{ss}} \\
\delta k_{ss} &= s_K y_{ss}.
\end{aligned}
$$

where $k_{ss} \equiv \frac{K_{ss}}{H_{ss}}$ and $y_{ss} = \frac{Y_{ss}}{H_{ss}}$.

Rewriting Production

- Recall that we had output per effective human capital $y_t \equiv \frac{Y_t}{H_t}$ and physical capital per effective human capital $k_t \equiv \frac{K_t}{H_t}$.
- We can rewrite our production function

$$Y_t = K_t^{\alpha}[(1 - s_H)H_t]^{1-\alpha}$$

$$\frac{Y_t}{H_t} = \frac{K_t^{\alpha}[(1 - s_H)H_t]^{1-\alpha}}{H_t}$$

$$\frac{Y_t}{H_t} = \left(\frac{K_t}{H_t}\right)^{\alpha}\left[(1 - s_H)\left(\frac{H_t}{H_t}\right)^{1-\alpha}\right]$$

$$y_t = (1 - s_H)^{1-\alpha}k_t^{\alpha}.$$

Steady State

- We had

$$\begin{aligned} \delta k_{ss} &= s_K y_{ss} \\ y_{ss} &= (1-s_H)^{1-\alpha} k_{ss}^{\alpha}. \end{aligned}$$

  We have two equations and two unknowns (k_{ss}, y_{ss}).

- Rearranging yields

$$\begin{aligned} \delta k_{ss} &= s_K y_{ss} \\ \delta k_{ss} &= s_K (1-s_H)^{1-\alpha} k_{ss}^{\alpha} \\ k_{ss}^{1-\alpha} &= \frac{s_K}{\delta}(1-s_H)^{1-\alpha} \\ k_{ss} &= (1-s_H)\left(\frac{s_K}{\delta}\right)^{\frac{1}{1-\alpha}}. \end{aligned}$$

Steady State

- We had

$$k_{ss} = (1 - s_H)\left(\frac{s_K}{\delta}\right)^{\frac{1}{1-\alpha}}$$
$$y_{ss} = (1 - s_H)^{1-\alpha} k_{ss}^{\alpha}.$$

- Combining, we get

$$y_{ss} = (1 - s_H)\left(\frac{s_K}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

- Note that output per effective human capital is strictly increasing in the share of time enhancing human capital.
- Does this mean the optimal share of workers in the production sector is 1?

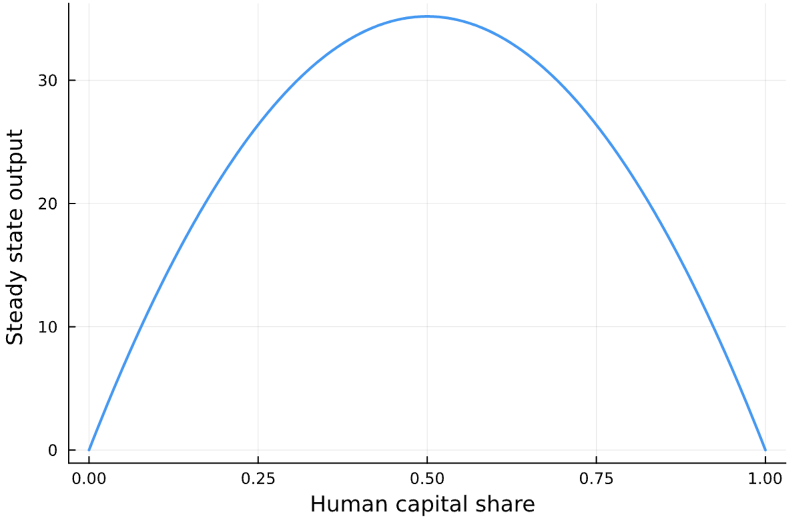
Steady State

- We can change our per human capital output y_{ss} to aggregate Y_{ss} by

$$\begin{aligned} Y_{ss} &= H_{ss} y_{ss} \\ &= \left(\frac{s_H^{\sigma}}{\delta}\right)^{\frac{1}{1-\sigma}} (1 - s_H) \left(\frac{s_K}{\delta}\right)^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

- Note that at different shares of time spent enhancing human capital (s_H), increasing s_H can have either positive or negative impacts on steady state output.

Optimal Share[2]



[2] $\sigma = 0.5,\ \delta = 0.10,\ s_K = 0.20,\ \alpha = 0.33$

Parameter Measurement

- ▷ We discussed earlier on the importance of ensuring each parameter in a model has a intuitive foundation.
- ▷ We've already discussed parameters s_K, δ, and α in previous lectures.
- ▷ We will focus on σ, which governs the intensity of human capital in education production and s_H, the share of time spent enhancing human capital.
- ▷ There are almost always multiple ways to estimate parameters in a model. For s_H, one could simply measure s_H directly.

Parameter Measurement

- To set s_H, we will choose a value such that the marginal return to physical capital is *equal* to the marginal return to human capital in the steady state.
- This gives us

$$\sigma(s_H H_{ss})^{\sigma-1} - \delta = \alpha K_{ss}^{\alpha-1}[(1-s_H)H_{ss}]^{1-\alpha} - \delta$$

- The idea here is that if the marginal return of one form of capital is higher than the other, then investments will be made in that form of capital until the marginal returns are equal.

Parameter Measurement

- We had

$$
\begin{aligned}
\sigma(s_H H_{ss})^{\sigma-1} - \delta &= \alpha K_{ss}^{\alpha-1}[(1-s_H)H_{ss}]^{1-\alpha} - \delta \\
\sigma \frac{(s_H H_{ss})^{\sigma}}{s_H H_{ss}} &= \alpha \frac{Y_{ss}}{K_{ss}}
\end{aligned}
$$

- In equilibrium we knew investment must equal depreciation, so

$$
\begin{aligned}
(s_H H_{ss})^{\sigma} &= \delta H_{ss} \\
s_K Y_{ss} &= \delta K_{ss}.
\end{aligned}
$$

- Using these gets us

$$
\sigma = \frac{\alpha s_H}{s_K}.
$$

Parameter Measurement

- We had

$$\sigma = \frac{\alpha s_H}{s_K}.$$

- Using capital share $\alpha = 1/3$ and investment rate $s_K = 20\%$, then we get

$$\sigma = \frac{5}{3} s_K.$$

- We'll set $s_H = \frac{1}{3}$ as an upper-bound, since the share of human capital in a country dedicated to training and building human capital is most certainly less than $1/3$.
- A suitable choice for the parameter σ would satisfy the condition $\sigma \leq \frac{5}{9}$.

Parameter Measurement

- Using different values of σ and estimates of investment in human capital, one finds that while important, human capital is insufficient in explaining differences across countries.
- Recall with our production and Solow models, we saw residual TFP was still very important in driving differences between countries.
- A natural question to ask is have we improved? That is, if we add human capital to a model similar to the Solow model, will the residual TFP be smaller?

Model: Adding Labor

- In the model above, we replaced workers with human capital for simplicity.
- We now will consider a model with physical capital, human capital, **and** labor, making it as similar as possible to our Solow model for comparison.
- Consider the following production function

$$Y_t = AK_t^{\alpha} H_t^{\beta} N_t^{1-\alpha-\beta}$$

  where α, β govern the contribution of capital and human capital to output lie between $0, 1$ non-inclusive.

Model: Adding Labor

- The physical capital and human capital accumulation equations will be given by

$$K_{t+1} = s_K Y_t + (1-\delta)K_t$$
$$H_{t+1} = s_H Y_t + (1-\delta)H_t$$

- Note that s_H now represents the share of output used to invest in human capital.
- We will assume constant population growth ($\frac{\Delta N_{t+1}}{N_t} = \bar{n}$) each period.

Model: Adding Labor

- Note that defining per capita variables $y_t \equiv \frac{Y_t}{N_t}$, $k_t \equiv \frac{K_t}{N_t}$ and $h_t \equiv \frac{H_t}{N_t}$, we can get

$$\begin{aligned}
\frac{Y_t}{N_t} &= \frac{AK_t^{\alpha}H_t^{\beta}N_t^{1-\alpha-\beta}}{N_t} \\
y_t &= A\left(\frac{K_t}{N_t}\right)^{\alpha}\left(\frac{H_t}{N_t}\right)^{\beta}\left(\frac{N_t}{N_t}\right)^{1-\alpha-\beta} \\
y_t &= Ak_t^{\alpha}h_t^{\beta}.
\end{aligned}$$

- Output per person is dependent on TFP (A), physical capital per person (k_t), and human capital per person (h_t).

Steady State

- Recall our capital accumulation equations

$$K_{t+1} = s_K Y_t + (1-\delta) K_t$$
$$H_{t+1} = s_H Y_t + (1-\delta) H_t.$$

- The physical capital accumulation equation is identical to the one from the Solow model. The same argument we used to transform these equations into per capita equations applies here and we get

$$\Delta k_{t+1} = s_K y_t - (\delta + \bar{n}) k_t$$
$$\Delta h_{t+1} = s_H y_t - (\delta + \bar{n}) h_t.$$

Steady State

- We can get steady state equations by

$$\begin{aligned} y_{ss} &= Ak_{ss}^{\alpha} h_{ss}^{\beta} \\ 0 &= s_K y_{ss} - (\delta + \bar{n}) k_{ss} \\ 0 &= s_H y_{ss} - (\delta + \bar{n}) h_{ss}. \end{aligned}$$

- With three variables and three equations, we can substitute and solve to get

$$y_{ss} = \left(A \frac{s_K^{\alpha} s_H^{\beta}}{(\delta + \bar{n})^{\alpha+\beta}} \right)^{\frac{1}{1-\alpha-\beta}} .$$

Model Comparison

- We found that

$$y_{ss} = \left(A \frac{s_K^{\alpha} s_H^{\beta}}{(\delta + \bar{n})^{\alpha+\beta}} \right)^{\frac{1}{1-\alpha-\beta}} .$$

- Recall that in the Solow model with *only* physical capital, we had

$$y_{ss} = \left(A \frac{s_K^{\alpha}}{(\delta + \bar{n})^{\alpha}} \right)^{\frac{1}{1-\alpha}} .$$

- We want to answer the question if adding human capital to the model helped decrease the residual left unexplained by the model.

Model Comparison

- Recall in our lecture on production we inverted the production function to get estimates for TFP (A).
- Suppose we were to look at steady state output and invert our equations from the previous slide to get estimates for TFP (A).
- We can then ask whether TFP, the residual unexplained by the model, is larger in the Solow model with or without human capital.

Model Comparison

- We had

$$y_{ss} = \left(A_H \frac{s_K^{\alpha} s_H^{\beta}}{(\delta + \bar{n})^{\alpha+\beta}} \right)^{\frac{1}{1-\alpha-\beta}}$$

$$y_{ss} = \left(A_K \frac{s_K^{\alpha}}{(\delta + \bar{n})^{\alpha}} \right)^{\frac{1}{1-\alpha}} .$$

  where A_H and A_K denote TFP with and without human capital respectively.

- Inverting these and solving for TFP (A) yields the fraction

$$\frac{A_H}{A_K} = y_{ss}^{-\beta} \left(\frac{\delta + \bar{n}}{s_H} \right)^{\beta} .$$

Model Comparison

- We had
$$\frac{A_H}{A_K} = y_{ss}^{-\beta} \left(\frac{\delta + \bar{n}}{s_H} \right)^{\beta}.$$
- Assuming $\beta > 0$ (human capital affects output), then $y_{ss}^{-\beta}$ will be very small while $\left(\frac{\delta+\bar{n}}{s_H} \right)^{\beta}$ will be close to 1.
- We should expect $\frac{A_H}{A_K} < 1$, which tells us the residual left unexplained is reduced when considering human capital.

Moving Forward

- Adding human capital to the Solow model added an element our Solow model did not capture and incorporated it through expanding our consideration of what is capital.
- Human capital is an important factor in explaining cross-country income differences, but not sufficient to explain the entire difference.
- Next time we will incorporate unmeasured capital into the Solow model.